Remarks on the determination of the Landau gauge OPE for the asymmetric three gluon vertex

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We compute a compact operator product expansion (OPE) formula describing power corrections to the perturbative expression for the asymmetric momentum subtraction–($\widetilde{\text{MOM}}$-)renormalized running coupling constant up to the leading logarithm. By the use of a phenomenological hypothesis leading to the factorization of the condensates through a perturbative vacuum insertion, the only relevant condensate in the game is $\langle A^2\rangle$. The validity of the OPE formula is tested by searching for a good-quality coherent description of previous lattice evaluations of the $\widetilde{\text{MOM}}$-renormalized gluon propagator and running coupling.

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That the running coupling constant can be extracted from the three-gluon vertex in the Landau gauge was proposed several years ago in a seminal work [1]. This method appears closer to a physical interpretation than Schrödinger functionals [2] but, and mainly for the same reason, more systematic effects should be included to produce a reliable prediction for the perturbative α_S. In particular, the first statistically meaningful attempts to follow that method missed the impact of power corrections [3] and failed to give an estimate for the coupling constant comparable to others in literature (that of Ref. [2], for instance). In parallel, these power corrections to the gluon propagator and to the running coupling constant have been greatly studied in recent years [4,5].

Although the first trials were rather inconclusive [6], momentum power contributions have been manifestly put in evidence [7,8] for the Landau gauge three-gluon coupling constant renormalized in both symmetric (MOM) and asymmetric ($\widetilde{\text{MOM}}$) momentum subtraction schemes. In Ref. [7] the parameter $\Lambda_{\overline{\text{MS}}}$ is estimated from the matching of $\widetilde{\text{MOM}}$ α_S lattice results to a perturbative three-loop formula corrected by an unavoidable $1/p^2$ term. This naive ansatz used in Ref. [7] seems to eliminate most of the systematic deviation from the three-gluon vertex estimate of the perturbative α_S and a precise prediction of $\Lambda_{\overline{\text{MS}}}$ emerges in full agreement with that of Ref. [2]. Unfortunately, the errors quoted in this work were clearly underestimated. In fact, the prediction of $\Lambda_{\overline{\text{MS}}}$ is so sensitive to a logarithmic dependence on the momentum scale of the coefficient of $1/p^2$ that it appears to range over an interval of 40 MeV [9].

On the other hand, in Refs. [8,9] a description in terms of operator product expansion (OPE) for the power corrections to the MOM three-gluon α_S was successfully tried consistently with an analogous description for the gluon propagator. The OPE approach provides through Shifman-Vainshtein-Zakharov (SVZ) factorization [10] a perturbative tool to obtain the leading logarithmic dependence of nontrivial Wilson coefficients. In particular, it can be applied to compute the coefficient of A^2 in the MOM case for the Landau gauge three-gluon coupling constant [9]. Obviously, the expectation value of A^2 in a nonperturbative vacuum is also estimated via this OPE approach. A non-gauge-invariant condensate, that, being in the Landau gauge, is invariant under infinitesimal gauge transformations and thus connected in some way to the gauge invariant $\langle A^2_{\text{min}}\rangle$ defined in [11], is then computed.

The same approach for $\widetilde{\text{MOM}}$ coupling constant requires use of

$$
\begin{aligned}
T^*(\widetilde{A}^a_\mu(-p)\widetilde{A}^b_\nu(p)) ={}& (c_0)^{ab}_{\mu\nu}(p)1+(c_1)^{ab\mu'}_{\mu\nu a'}(p):A^{a'}_{\mu'}(0):\\
&+(c_2)^{ab\mu'\nu'}_{\mu\nu a'b'}(p):A^{a'}_{\mu'}(0)A^{b'}_{\nu'}(0):\\
&+(c_3)^{ab\mu'\nu'\rho'}_{\mu\nu a'b'c'}(p):A^{a'}_{\mu'}(0)A^{b'}_{\nu'}(0)\\
&\times A^{c'}_{\rho'}(0):+\cdots, \qquad (1)
\end{aligned}
$$

where the expansion to the three-gluon local operator is necessary because the three-point Green function in $\widetilde{\text{MOM}}$ can be written as

$$
\begin{aligned}
&\langle T^*(\widetilde{A}^a_\mu(-p)\widetilde{A}^b_\nu(p)\widetilde{A}^c_\rho(0))\rangle_{\text{NP}}\\
&\quad\equiv\langle 0|T^*(\widetilde{A}^a_\mu(-p)\widetilde{A}^b_\nu(p))|g^c_\rho\rangle_{\text{NP}}\\
&\quad=(c_1)^{ab\mu'}_{\mu\nu a'}\langle 0|:A^{a'}_{\mu'}(0):|g^c_\rho\rangle_{\text{NP}}\\
&\qquad+(c_3)^{ab\mu'\nu'\rho'}_{\mu\nu a'b'c'}\langle 0|:A^{a'}_{\mu'}(0)A^{b'}_{\nu'}(0)A^{c'}_{\rho'}(0):|g^c_\rho\rangle_{\text{NP}}\\
&\qquad+\cdots. \qquad (2)
\end{aligned}
$$

The index NP refers to the nonperturbative nature of the vacuum state in Eq. (2), while T^* refers to the standard time ordered product in momentum space. The following points should be noticed. (i) No other local operators with the same dimension of A^2 are written in Eq. (1) because, unlike the identity or A^2 itself, they do not generate a non-null vacuum expectation value.[1] (ii) Operators other than the three-gluon

[1]No Lorentz invariant tensor with an odd number of indices can

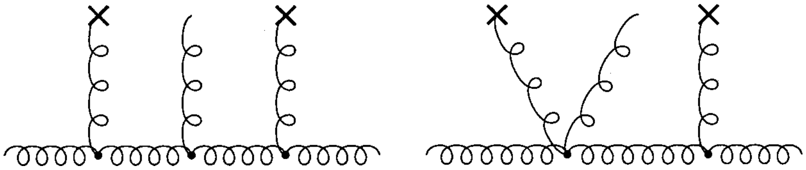

FIG. 1. Diagrams contributing to the tree-level Wilson coefficient in Eq. (7) Crosses mark the soft-gluon legs coming from the condensate.

local one for the same dimension, such as $\partial_\mu A^b_\nu(0) A^c_\rho(0)$, do not appear explicitly because they are phenomenologically supposed not to survive, as will be argued below. The identity and A^2 clearly do not contribute to the matrix element considered in Eq. (2).

Following standard SVZ techniques to obtain the peturbative expansion of OPE Wilson coefficients, we compute the appropriate matrix element of Eq. (1)'s left-hand side (LHS) to the wanted order. It is obvious that taking the perturbative vacuum leads to

$$\langle 0|T^*(\widetilde{A}^a_\mu(-p)\widetilde{A}^b_\nu(p)\widetilde{A}^c_\rho(0))|0\rangle = (c_1)^{ab\mu'}_{\mu\nu a'}\widetilde{G^{(2)}}^{a'c}_{\mu'\rho}(0), \tag{3}$$

where c_1 can be straightforwardly identified with the perturbative expansion for the Green function with an amputated soft-gluon leg,

$$\Gamma^{ab\mu'}_{\mu\nu a'}(p,-p,0) \equiv -2p^{\mu'} g^{\perp}_{\mu\nu}(p) f^{ab}_{a'} G^{(3)}_{\mathrm{pert}}(p^2), \tag{4}$$

which should be proportional to its Landau gauge tree-level tensor. Beyond this purely perturbative first contribution, the situation is slightly more complicated. We consider now the matrix element on the LHS of Eq. (2) between two external soft gluons and the perturbative vacuum. Then, we obtain up to the considered order in perturbation theory

$$\begin{aligned}&\langle 0|T^*(\widetilde{A}^a_\mu(-p)\widetilde{A}^b_\nu(p)\widetilde{A}^c_\rho(0))|g^l_\lambda g^s_\sigma\rangle_{\mathrm{connected}}\\ &\quad = (c_3)^{ab\mu'\nu'\rho'}_{\mu\nu a'b'c'}(p)\langle 0|{:}A^{a'}_{\mu'}(0)A^{b'}_{\nu'}(0)A^{c'}_{\rho'}(0){:}|g^c_\rho g^l_\lambda g^s_\sigma\rangle.\end{aligned} \tag{5}$$

For the matrix element on the right hand side (RHS) we have

$$\begin{aligned}&\langle 0|{:}A^{a'}_{\mu'}(0)A^{b'}_{\nu'}(0)A^{c'}_{\rho'}(0){:}|g^c_\rho g^l_\lambda g^s_\sigma\rangle\\ &\quad=\widetilde{G^{(2)}}^{ss'}_{\sigma\sigma'}(0)\widetilde{G^{(2)}}^{ll'}_{\lambda\lambda'}(0)\widetilde{G^{(2)}}^{ct}_{\rho\tau}(0)\\ &\quad\quad\times\{\mathcal{P}^{s'l't}_{\sigma'\lambda'\tau} g^{\sigma'}_{\mu'} g^{\lambda'}_{\nu'} g^{\tau}_{\rho'} \delta^{a'}_{s'}\delta^{b'}_{l'}\delta^{c'}_{t} + O(\alpha)\},\end{aligned} \tag{6}$$

where $\mathcal{P}$ refers to all the possible permutations of the couples $(\sigma' s')$, $(\lambda' l')$, and (τt). The Wilson coefficient c_3 may thus be computed at tree-level order as

$$\mathcal{P}^{s'l't}_{\sigma'\lambda'\tau}(c_3)^{ab\sigma'\lambda'\tau}_{\mu\nu s'l't} = \frac{\langle \widetilde{A}^l_\lambda(0)\widetilde{A}^a_\mu(-p)\widetilde{A}^b_\nu(p)\widetilde{A}^c_\rho(0)\widetilde{A}^s_\sigma(0)\rangle}{\widetilde{G^{(2)}}^{ss'}_{\sigma\sigma'}(0)\widetilde{G^{(2)}}^{ll'}_{\lambda\lambda'}(0)\widetilde{G^{(2)}}^{ct}_{\rho\tau}(0)}. \tag{7}$$

The ratio in Eq. (7) represents symbolically all the tree-level diagrams with five gluon legs where the three carrying zero momentum are cut (see Fig. 1).

Had we directly dealt with the three-gluon local operator ${:}A^{a'}_{\mu'}(0)A^{b'}_{\nu'}(0)A^{c'}_{\rho'}(0){:}$, the evaluation of the higher order tensors involved would require a much more tedious calculation. On the other hand, *vacuum insertion* between one gluon field and the other two leads to

$$\begin{aligned}&[\langle 0|{:}A^{a'}_{\mu'}(0)A^{b'}_{\nu'}(0)A^{c'}_{\rho'}(0){:}|g^c_\rho\rangle_{\mathrm{NP}}]_{\mathrm{R},\mu}\\ &\quad=\frac{\langle A^2\rangle_{\mathrm{R},\mu}\widetilde{G}^{tc}_{\tau\rho}(0,\mu^2)}{4(N_c^2-1)}\ \mathcal{T}^{a'b'c'\tau}_{\mu'\nu'\rho't}\end{aligned} \tag{8}$$

with

$$\begin{aligned}\mathcal{T}^{a'b'c'\tau}_{\mu'\nu'\rho't} &= g_{\mu'\nu'}\delta^{a'b'}g^{\tau}_{\rho'}\delta^{c'}_{t} + g_{\mu'\rho'}\delta^{a'c'}g^{\nu'}_{\tau}\delta^{b'}_{t}\\ &\quad + g_{\rho'\nu'}\delta^{c'b'}g^{\tau}_{\mu'}\delta^{a'}_{t}.\end{aligned} \tag{9}$$

We may phenomenologically assume vacuum insertion to work for a certain renormalization momentum scale. Thus, the replacement

$${:}A^{a'}_{\mu'}(0)A^{b'}_{\nu'}(0)A^{c'}_{\rho'}(0){:} \to \frac{{:}A^2 A^t_\tau(0){:}}{4(N_c^2-1)}\mathcal{T}^{a'b'c'\tau}_{\mu'\nu'\rho't} \tag{10}$$

can be done for Eq. (2), the other components of the tensor on the LHS not being required for our purposes. Furthermore, the local gluon field $A^t_\tau(0)$ is to be contracted with the zero-momentum gluon field defined for the vertex. The same vacuum insertion assumption leads us to argue that, for instance,

$$\begin{aligned}&[\langle 0|{:}\partial'_\mu A^{b'}_{\nu'}(0)A^{c'}_{\rho'}(0){:}|g^c_\rho\rangle_{\mathrm{NP}}]_{\mathrm{R},\mu}\\ &\quad\to\langle \partial'_\mu A^{b'}_{\nu'}(0)\rangle_{\mathrm{R},\mu}\widetilde{G}^{c'c}_{\rho'\rho}(0,\mu^2) = 0.\end{aligned} \tag{11}$$

Thus, the only nonzero surviving condensate comes from the three-gluon local operator.

It is easy to see that, using Eq. (10), the relevant coefficient multiplying the local operator in Eq. (1) is

$$\begin{aligned}&(c_3)^{ab\mu'\nu'\rho'}_{\mu\nu a'b'c'}(p)\mathcal{T}^{a'b'c'\tau}_{\mu'\nu'\rho't}\\ &\quad=\frac{1}{2}\frac{\langle \widetilde{A}^l_\lambda(0)\widetilde{A}^a_\mu(-p)\widetilde{A}^b_\nu(p)\widetilde{A}^c_\rho(0)\widetilde{A}^s_\sigma(0)\rangle}{\widetilde{G^{(2)}}^{sa'}_{\sigma\mu'}(0)\widetilde{G^{(2)}}^{lb'}_{\lambda\nu'}(0)\widetilde{G^{(2)}}^{ct}_{\rho\tau}(0)}\ g_{\mu'\nu'}\delta^{a'b'}\end{aligned} \tag{12}$$

be built without nonzero momenta.

where the RHS may be straightforwardly obtained from Eq. (7). This last Eq. (12) gives the prescription in which Lorentz and color indices of external gluon legs are contracted in the diagrams contributing to the tree-level Wilson coefficient (see Fig. 1).

Thus, since Eq. (4)'s is the only Landau gauge tensor for the asymmetric three-gluon vertex [3], we can write, for $p^2 = -k^2$,

$$
\begin{aligned}
k^4 \langle T^*(\widetilde{A}^a_\mu(-p)\widetilde{A}^b_\nu(p)\widetilde{A}^c_\rho(0))\rangle_{\rm NP} &= -2p^\tau g^\perp_{\mu\nu}(p) f^{ab}_t \left[c_1\left(g,\frac{k^2}{\Lambda^2}\right)\widetilde{G^{(2)}}{}^{tc}_{\tau\rho}(0) \right. \\
&\left. + c_3\left(g,\frac{k^2}{\Lambda^2}\right)\frac{\langle 0|:A^2A^t_\tau(0):|g^c_\rho\rangle_{\rm NP}}{4(N_c^2-1)}\frac{1}{-k^2}\right]. \qquad (13)
\end{aligned}
$$

We know the scalar coefficients c_1, c_3 in Eq. (13) to be dimensionless by OPE power counting rules, and hence both depend only on the bare coupling g and the dimensionless ratio of momentum to regularization scale.[2] The term c_1 can be obtained from Eq. (3) and, at tree level, c_3 is to be computed by projecting over the Landau gauge tree-level tensor in Eq. (4) the result from Eq. (12),

$$
\begin{aligned}
c_{3,tree\ level}\left(g,\frac{k^2}{\Lambda^2}\right) &= (c_3)^{ab\mu'\nu'\rho'}_{\mu\nu a'b'c'}(p)\mathcal{T}^{a'b'c'\tau}_{\mu'\nu'\rho't} \\
&\times \frac{1}{-6N_C(N_C-1)k^2} p_\tau g^{\perp\mu\nu}(p) f_{ab}{}^t \\
&= 3g. \qquad (14)
\end{aligned}
$$

If we renormalize following the $\widetilde{\rm MOM}$ prescription at momentum scale μ^2, and then apply the assumed vacuum insertion factorization, we can write

$$
\begin{aligned}
k^4[\langle T^*(\widetilde{A}^a_\mu(-p)\widetilde{A}^b_\nu(p)\widetilde{A}^c_\rho(0))\rangle]_{{\rm R},\mu^2} &= -2p^\tau g^\perp_{\mu\nu}(p) f^{ab}_t \widetilde{G^{(2)}}{}^{tc}_{\tau\rho}(0,\mu^2)\left[c_1\left(\frac{k^2}{\mu^2},\alpha(\mu)\right)\right. \\
&\left. + c_3\left(\frac{k^2}{\mu^2},\alpha(\mu)\right)\frac{\langle A^2\rangle_{{\rm R},\mu^2}}{4(N_c^2-1)}\frac{1}{-k^2}\right], \qquad (15)
\end{aligned}
$$

where, after renormalization, the Wilson coefficients depend on the ratio of the momentum and the renormalization scale, and on $\alpha(\mu) = g_R^2(\mu)/(4\pi)$, i.e., the coupling constant consistently renormalized in $\widetilde{\rm MOM}$.

In the $\widetilde{\rm MOM}$ scheme, the renormalized Green functions take formally, at the renormalization scale, the same tree-level value but in terms of the renormalized coupling constant instead of the bare one. This defines $Z^{\rm MOM}(\mu) = \mu^2 G^{(2)}(\mu^2)$ to renormalize appropriately the two-point Green function, where $G^{(2)}$ is the scalar factor of the bare two-point Green function defined in Ref. [3]. The three-gluon Green function is then renormalized by dividing by $[Z^{\rm MOM}(\mu)]^{3/2}$. One gets

$$
g_R(k^2) = k^4 \frac{G_R^{(3)}(k^2,\mu^2)}{G_R^{(2)}(0,\mu^2)}[k^2 G_R^{(2)}(k^2,\mu^2)]^{-1/2}. \qquad (16)
$$

The renormalized scalar factor for the three-gluon Green function with an amputated soft gluon, $G_R^{(3)}/G_R^{(2)}$, can be projected out from Eq. (15), as is done for c_3 in Eq. (14), while for $G_R^{(2)}(k^2,\mu^2)$ we can write

$$
\begin{aligned}
k^2 G_R^{(2)}(k^2,\mu^2) &= c_0\left(\frac{k^2}{\mu^2},\alpha(\mu)\right) \\
&+ c_2\left(\frac{k^2}{\mu^2},\alpha(\mu)\right)\frac{\langle A^2\rangle_{{\rm R},\mu^2}}{4(N_c^2-1)}\frac{1}{-k^2}, \qquad (17)
\end{aligned}
$$

where the scalar coefficients c_0 and c_2 can be derived from those in Eq. (1) and computed similarly to c_1 and c_3, as explained in Ref. [9]. Thus, after replacing in Eq. (16) we will finally get

$$
\begin{aligned}
g_R(k^2) &= c_1\left(\frac{k^2}{\mu^2},\alpha(\mu)\right)\left[c_0\left(\frac{k^2}{\mu^2},\alpha(\mu)\right)\right]^{-1/2} \\
&\times\left[1+\left(\frac{c_3(k^2/\mu^2,\alpha(\mu))}{c_1(k^2/\mu^2,\alpha(\mu))}\right.\right. \\
&\left.\left.-\frac{1}{2}\frac{c_2(k^2/\mu^2,\alpha(\mu))}{c_0(k^2/\mu^2,\alpha(\mu))}\right)\frac{\langle A^2\rangle_{{\rm R},\mu^2}}{4(N_c^2-1)}\frac{1}{-k^2}\right]. \qquad (18)
\end{aligned}
$$

The purpose is now to compute to leading logarithms the subleading Wilson coefficients in Eq. (15), as was done in Ref. [9]. To this end, it will be useful to consider the following operator expansion:

$$
\begin{aligned}
&\frac{2}{9k^2}p_\tau g^{\perp\mu\nu}(p) f^t_{ab}[T^*(\widetilde{A}^a_\mu(-p)\widetilde{A}^b_\nu(p)\widetilde{A}^c_\rho(0))]_{{\rm R},\mu^2} \\
&= \frac{c_3(k^2/\mu^2,\alpha(\mu))}{-k^6}[:A^2A^t_\tau(0):\widetilde{A}^c_\rho(0)]_{{\rm R},\mu} + \cdots; \qquad (19)
\end{aligned}
$$

where the ellipsis refers to terms with powers of $1/k$ other than 6. If the vacuum expectation in the nonperturbative vacuum is considered for the RHS of Eq. (19), the result under the vacuum insertion hypothesis is known to be diagonal in color and Lorentz spaces. Then we contract on both sides of Eq. (19) with $g^{\rho\tau}\delta_{ct}$ and take the following matrix element (see Ref. [12]):

[2] The dependence on the regularization momentum scale (a^{-1} in lattice regularization or $\epsilon^{-1}\mu$ in dimensional, for instance) has been omitted up to now to simplify the notation.

$$\begin{aligned}&\frac{2}{9k^2}p^\rho g^{\perp\mu\nu}(p)f_{abc}\langle 0|T^*(\tilde{A}^a_\mu(-p)\tilde{A}^b_\nu(p)\tilde{A}^c_\rho(0))|g^s_\sigma g^l_\lambda\rangle_{\mathrm{R},\mu^2}\\&\quad=\frac{c_3(k^2/\mu^2,\alpha(\mu))}{-k^6}\langle 0|{:}A^2A^\rho_c(0){:}|g^c_\rho g^s_\sigma g^l_\lambda\rangle_{\mathrm{R},\mu^2}+\cdots,\end{aligned}\tag{20}$$

where we take external gluons in a perturbative vacuum and carrying soft momenta. From Eq. (20) we get

$$\begin{aligned}&-\frac{2}{9}k^4\frac{p^\rho g^{\perp\mu\nu}(p)f_{abc}\langle 0|T^*(\tilde{A}^a_\mu(-p)\tilde{A}^b_\nu(p)\tilde{A}^c_\rho(0))|g^s_\sigma g^l_\lambda\rangle}{\langle 0|{:}A^2A^\rho_c(0){:}|g^c_\rho g^s_\sigma g^l_\lambda\rangle}\\&\quad=Z_3(\mu^2)Z^{-1}_{A^3}(\mu^2)c_3\left(\frac{k^2}{\mu^2},\alpha(\mu)\right)\\&\quad=Z_3^{-1/2}(\mu^2)\overline{Z}^{-1}c_3\left(\frac{k^2}{\mu^2},\alpha(\mu)\right),\end{aligned}\tag{21}$$

where, always in $\widetilde{\mathrm{MOM}}$ prescription, $\tilde{A}_R=Z_3^{-1/2}\tilde{A}$ and Z_{A^3} is defined such that

$$\begin{aligned}&[{:}A^2A^\rho_c(0){:}\tilde{A}^c_\rho(0)]_{\mathrm{R},\mu^2}\\&\quad=Z^{-1}_{A^3}(\mu^2)Z_3^{-1/2}(\mu^2){:}A^2A^\rho_c(0){:}\tilde{A}^c_\rho(0),\end{aligned}\tag{22}$$

while $\overline{Z}\equiv Z_{A^3}Z_3^{-3/2}$ is a useful notation for the constant renormalizing the matrix element for the three-gluon local operator where the external soft gluons are explicitly cut. If we recover the divergent factor $\hat{Z}\equiv Z_{A^2}Z_3^{-1}$ introduced in Ref. [9] for the matrix element of the two-gluon local operator coming from proper vertex corrections, $\overline{Z}$ can be decomposed as

$$\overline{Z}(\mu^2)\equiv\hat{Z}(\mu^2)Z_\kappa(\mu^2),\tag{23}$$

where $\hat{Z}$ takes the divergent part coming from the diagrams for the matrix element on the (RHS) of Eq. (20) which can be factorized [diagrams (a,b) of Fig. 2] as in Eq. (8), and Z_κ should be computed from those that cannot [diagrams (c,d) of Fig. 2]. Then, taking the logarithmic derivatives with respect to μ on both sides of Eq. (21), we get the following renormalization group differential equation:

$$\begin{aligned}&\left\{-2\overline{\gamma}[\alpha(\mu)]-\gamma[\alpha(\mu)]+\frac{\partial}{\partial\ln\mu}\right.\\&\quad\left.+\beta[\alpha(\mu)]\frac{\partial}{\partial\alpha}\right\}c_3\left(\frac{k^2}{\mu^2},\alpha(\mu)\right)=0,\end{aligned}\tag{24}$$

with the formal solution (see Ref. [12])

$$c_3\left(\frac{k^2}{\mu^2},\alpha(\mu)\right)=c_3(1,\alpha(k))\left(\frac{\alpha(k)}{\alpha(\mu)}\right)^{-(2\overline{\gamma}_0+\gamma_0)/2\beta_0},\tag{25}$$

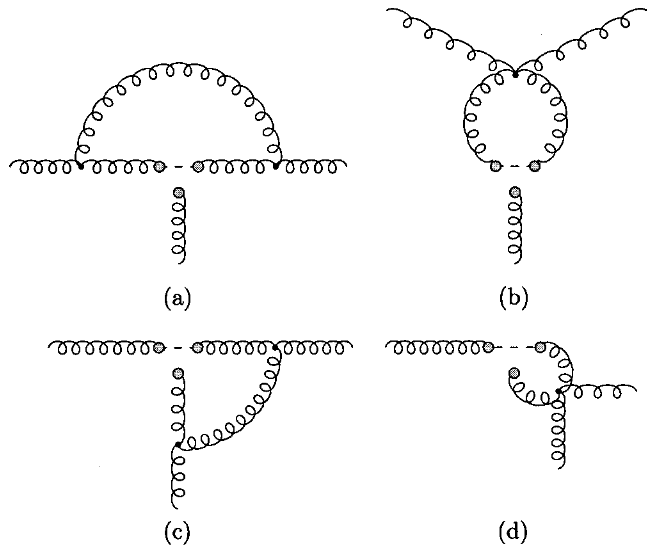

FIG. 2. All the possible leg permutations from diagrams in the figure contribute to the renormalization constant $\overline{Z}$ defined in the text. (a)- and (b)-like diagrams, which do not break the assumed-to-work factorization hypothesis, are renormalized by $\hat{Z}$ previously computed in [9]. (c) and (d) factorization breaking diagrams give Z_κ. The local operators are drawn as gray bullets; the two joined by a dashed line represent the ones contracted to give A^2 in Eqs. (19)–(22) through replacement in Eq. (10)

where

$$\overline{\gamma}[\alpha(\mu)]=\frac{d}{d\ln\mu^2}\ln\overline{Z}(\mu^2)=-\overline{\gamma}_0\frac{\alpha(\mu)}{4\pi}+\cdots,$$

$$\begin{aligned}\gamma[\alpha(\mu)]&=\frac{d}{d\ln\mu^2}\ln Z_3(\mu^2)\\&=-\left[\gamma_0\frac{\alpha(\mu)}{4\pi}+\gamma_1\left(\frac{\alpha(\mu)}{4\pi}\right)^2\right.\\&\quad\left.+\gamma_2\left(\frac{\alpha(\mu)}{4\pi}\right)^3+\cdots\right],\end{aligned}$$

$$\begin{aligned}\beta[\alpha(\mu)]&=\frac{d}{d\ln\mu}\alpha(\mu)\\&=-\left(\frac{\beta_0}{2\pi}\alpha^2(\mu)+\frac{\beta_1}{4\pi^2}\alpha^3(\mu)\right.\\&\quad\left.+\frac{\beta_2}{(4\pi)^3}\alpha^4(\mu)+\cdots\right).\end{aligned}\tag{26}$$

The boundary condition of Eq. (24) is given by our $\widetilde{\mathrm{MOM}}$-like prescription for the renormalization of the condensate by Z_{A^3}: the condensate is renormalized such that the Wilson coefficient takes the tree-level form at the renormalization point. Thus the prefactor $c_3(1,\alpha(k))$ has to be matched at tree level to Eq. (14), and the only solution to the leading logarithm is

$$c_3(1,\alpha(k))=3[g_R(k)]^3\left[1+\mathcal{O}\left(\frac{1}{\log(k/\Lambda_{\rm QCD})}\right)\right]. \quad (27)$$

Identifying the non-power-corrected term of Eq. (18) with the purely perturbative coupling constant, $c_1(k^2/\mu^2,\alpha(\mu))$ is known to be

$$c_1\left(\frac{k^2}{\mu^2},\alpha(\mu)\right)=g_{\rm R,pert}(k^2)\left[c_0\left(\frac{k^2}{\mu^2},\alpha(\mu)\right)\right]^{1/2}. \quad (28)$$

Thus, we take from Ref. [9] the following leading order results:

$$c_0\left(\frac{k^2}{\mu^2},\alpha(\mu)\right)=\left(\frac{\alpha(k)}{\alpha(\mu)}\right)^{\gamma_0/\beta_0}$$

$$c_2\left(\frac{k^2}{\mu^2},\alpha(\mu)\right)=3g_R^2(k^2)\left(\frac{\alpha(k)}{\alpha(\mu)}\right)^{-\hat{\gamma}_0/\beta_0}, \quad (29)$$

where, as usual, $\hat{\gamma}_0$ is defined by

$$\frac{d}{d\ln\mu^2}\ln\hat{Z}(\mu^2)=-\hat{\gamma}_0\frac{\alpha(\mu)}{4\pi}+\cdots. \quad (30)$$

Equation (18) can then be applied to obtain $\alpha_{\widetilde{\rm MOM}}=g_R^2/(4\pi)$ which, after the appropriate Wick rotation, takes the following form in Euclidean space:

$$\alpha_{\widetilde{\rm MOM}}(k^2)=\alpha_{\rm pert}(k^2)\left\{1+\frac{T(\mu)}{k^2}\left[\ln\left(\frac{k}{\Lambda}\right)\right]^{(\hat{\gamma}_0+\gamma_0)/\beta_0-1}\right.$$
$$\left.\times\left(2\left[\frac{\ln\left(\frac{k}{\Lambda}\right)}{\ln(\mu/\Lambda)}\right]^{\kappa_0/\beta_0}-1\right)\right\} \quad (31)$$

with

$$T(\mu)=\frac{6\pi^2}{\beta_0}\frac{\langle A^2\rangle_{R,\mu}}{N_C^2-1}\left[\ln\left(\frac{\mu}{\Lambda}\right)\right]^{-(\hat{\gamma}_0+\gamma_0)/\beta_0}, \quad (32)$$

$\langle A^2\rangle_{R,\mu}$ being now the Euclidean condensate; and

$$\kappa_0=\overline{\gamma}_0-\hat{\gamma}_0, \quad (33)$$

as immediately follows from Eq. (23) if κ_0 is defined from

$$\frac{d}{d\ln\mu^2}\ln\hat{Z}_\kappa(\mu^2)=-\kappa_0\frac{\alpha(\mu)}{4\pi}+\cdots. \quad (34)$$

The perturbative coefficients for the flavorless case,

$$\beta_0=11,\quad \beta_1=51,\quad \gamma_0=\frac{13}{2}, \quad (35)$$

are universal, while for the coefficients in the $\widetilde{\rm MOM}$ scheme, β_2 was computed in [3], and γ_1 and γ_2 in [13]:

$$\beta_2\simeq4824,\quad \gamma_1=\frac{29}{8},\quad \gamma_2=960. \quad (36)$$

In a recent paper [9] we computed $\hat{\gamma}_0$ from diagrams identical to these in Figs. 2(a) and 2(b); κ_0, defined in Eq. (33) can be computed from diagrams (c) and (d) in Fig. 2 to give

$$\hat{\gamma}_0=\frac{3N_C}{4},\quad \kappa_0=-\frac{9N_C}{136}. \quad (37)$$

We proceeded to evaluate the diagrams for κ_0 in total analogy with the calculation of those for $\hat{\gamma}_0$. For instance, we made the diagrams infrared safe by considering a momentum flow incoming to the local operator. The details of the procedure can be found in Ref. [9]. It should be noticed that $\kappa_0\ll\hat{\gamma}_0$ and that, in practice, $\kappa_0/\beta_0\simeq0$ works as a good approximation to simplify Eq. (31). In other words, the scheme given by vacuum insertion factorization reveals itself to be coherent: *leading logarithm corrections violating the factorization induce a very small running with the renormalization scale for the factorized tree-level Wilson coefficient*.

Nevertheless, it is important to prove the assumption to work in order to obtain a good estimate of the Wilson coefficient for the asymmetric three-gluon Green function. To examine this question, we have performed the same combined fits shown in Ref. [9] for two- and three-point Green functions, at three loops for leading Wilson coefficients, to match lattice data taken from Refs. [3,13]. The Euclidean OPE formula for the two-point Green function is, from Ref. [9],

$$Z_{\rm Latt}^{\rm MOM}(k^2,a)=Z_{\rm Latt}^{\rm MOM}(\mu^2,a)c_0\left(\frac{k^2}{\mu^2},\alpha(\mu)\right)$$
$$\times\left[1+R(\mu)\left(\ln\frac{k}{\Lambda}\right)^{(\gamma_0+\hat{\gamma}_0)/\beta_0-1}\frac{1}{k^2}\right], \quad (38)$$

where

$$\frac{Z_{\rm Latt}^{\rm MOM}(k^2,a)}{Z_{\rm Latt}^{\rm MOM}(\mu^2,a)}=k^2G_R^{(2)}(k^2,\mu^2)+O(a^2) \quad (39)$$

and

$$R(\mu)=\frac{6\pi^2}{\beta_0(N_c^2-1)}\left(\ln\frac{\mu}{\Lambda}\right)^{-\gamma_0+\hat{\gamma}_0/\beta_0}\langle A^2\rangle_{\rm R,\mu}. \quad (40)$$

The coefficient $c_0(k^2/\mu^2,\alpha(\mu))$ is taken now to be expanded at three loops in terms of the $\widetilde{\rm MOM}$ scheme $\alpha(k)$; i.e., it will satisfy the same differential equation as $\gamma(\alpha)$ in Eq. (26) with a boundary condition that is apparent from Eq. (29). In the following, all the scale-dependent quantities will be shown at $\mu=10$ GeV. Furthermore, we have checked that

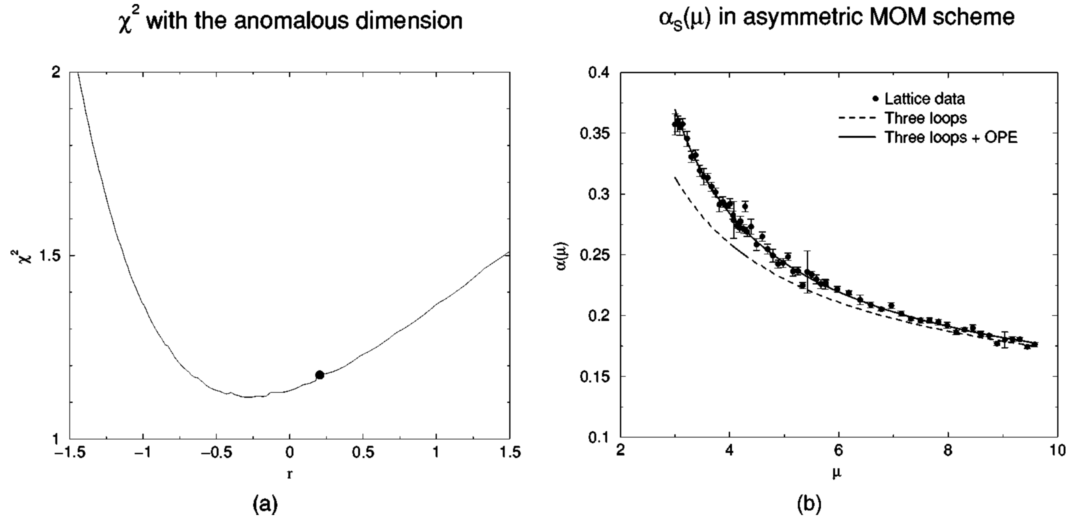

FIG. 3. (a) Quality of the fit as a function of the exponent $r=1-(\hat{\gamma}_0+\gamma_0)/\beta_0$ in Eq. (31). The dot stands for the value of r computed in this paper. In (b) is shown the fit of our lattice data for $\alpha_{\widetilde{MOM}}$ at three loops with the calculated anomalous dimension.

indeed neither the ratio of gluon condensate estimates nor $\Lambda_{\overline{\rm MS}}$ depends on this last momentum scale. The quality of the fits as a function of the free exponent of $\ln(k/\Lambda)$ in Eq. (31) has been explored and the results are shown in Fig. 3(a). We can conclude from these results that the approach given by vacuum insertion factorization provides a good estimate of this exponent.

The results for two particular values of the exponent r in Fig. 3(a) are of interest for the sake of comparison. The case $r=-1$ corresponds to the formula proposed in Ref. [7] to be matched to the lattice data: a perturbative three-loop formula + a term c/p^2, c being a constant. On the other hand, had we neglected the leading logarithm contributions $\hat{\gamma}_0=\gamma_0=0$, the exponent would be $r=1$. It can be seen from Fig. 3 that both values of r generate rather less good fits to the lattice data.

Then, Eqs. (31), (32) can be used to perform fits at two and three loops for the leading Wilson coefficients in order to estimate, from the asymmetric three-gluon Green function, the gluon condensate. The results of such fits, plotted in Fig. 3(b), are

$$\frac{\{\sqrt{\langle A^2\rangle_{R,\mu}}\}_{alpha}}{\{\sqrt{\langle A^2\rangle_{R,\mu}}\}_{prop}}=3.65(4), \quad \Lambda_{\overline{\rm MS}}=283(15)\ \text{MeV},$$
$$\chi^2=1.95 \tag{41}$$

for the two-loop fit, and

$$\frac{\{\sqrt{\langle A^2\rangle_{R,\mu}}\}_{alpha}}{\{\sqrt{\langle A^2\rangle_{R,\mu}}\}_{prop}}=1.7(3), \quad \Lambda_{\overline{\rm MS}}=260(18)\ \text{MeV},$$
$$\chi^2=1.18 \tag{42}$$

for the three-loop one.

The impressive improvement from two to three loops suggests that the approach presented in this work permits a reasonable approximation to the Wilson coefficient. The ratio decreases to almost σ away from 1 and both estimates of $\Lambda_{\overline{\rm MS}}$ and of the gluon condensate turn out to be close to the previous estimates obtained from the symmetric three-gluon Green function in Ref. [9] (see Table I). The scheme in this work and that of Ref. [9] differ only by the kinematics of the renormalization point. Such a different renormalization for the Green functions implies a different renormalization of the gluon condensate. However, the discrepancy for estimates of $\langle A^2\rangle_R$ in the two works is expected not to be important,[3] as indeed can be seen in Table I. A comparison of these estimates, mainly the ones obtained from the gluon propagator, strongly supports the claimed rather large contribution from the A^2 condensate [7–9] which might be connected to with the tachyonic gluon mass scale studied in Ref. [11].

A negative hint regarding previous results from the symmetric three-point Green function is nevertheless the higher central value of the ratio in Eq. 42 (1.2 in Ref. [9]). In principle two possible sources of discrepancies can be expected: either three loops is still insufficiently accurate for the estimate of the perturbative part in the $\widetilde{\rm MOM}$ renormalization scheme, or there is a deviation from the assumed vacuum insertion (or factorization) approximation. Both effects would have a direct impact on the bigger ratio we obtain for the asymmetric $\widetilde{\rm MOM}$ scheme. The very good agreement between the gluon condensates estimated from the gluon propagator previously discussed seems to point to factorization breaking as the major contributing factor. Still, the ratio in Eq. (42) is only about 2σ from 1, which is in our opinion a rather encouraging result. The $\langle A^2\rangle$ deduced from the propagator is in fair agreement with previous estimates, even though it is biased by the factorization hypothesis through the fit of $\Lambda_{\overline{\rm MS}}$, which combines the propagator and α_S. This good result of the propagator, as well as Fig. 3(a),

[3]To estimate the discrepancy for the nonperturbative estimates of $\langle A^2\rangle_R$ we need to compute beyond leading logarithm corrections, which is beyond the scope of this work.

TABLE I. Comparison between results obtained for the three-loop fit in the present work and in a previous one [9].

	This work	Symmetric three point
$\Lambda_{\overline{\mathrm{MS}}}$	260(18) MeV	233(28) MeV
$\{\sqrt{\langle A^2\rangle_{R,\mu}}\}_{prop}$	1.39(14) GeV	1.55(17) GeV
$\{\sqrt{\langle A^2\rangle_{R,\mu}}\}_{alpha}$	2.3(6) GeV	1.9(3) GeV

suggest that our formula describing the power corrections to α_S up to the leading logarithm yields a good approximation of the exact one.

A two-sided goal is thus achieved.

(i) The results of Ref. [9] turn out to be confirmed by the use of a slightly different renormalization scheme.

(ii) Vacuum insertion factorization applied to condensates playing the role of the OPE for the asymmetric three-gluon Green function results in a compact prediction for its OPE power corrections. The coefficient of the power correction has been computed to the leading logarithm, and thus a most important source of systematic uncertainty for the estimate of $\Lambda_{\overline{\mathrm{MS}}}$ in Ref. [7] is eliminated. The latter is a positive feature because the β function is perturbatively known at four loops in the asymmetric $\widetilde{\mathrm{MOM}}$ [14] and lattice evaluations, on the other hand, turn out to be statistically more precise in this last renormalization scheme.

A last consequence of this work and those from Refs. [8,9]: together they lead to the conclusion that the Green function methods, the three-gluon vertex in particular, provide us with a reliable and precise enough estimate for the running coupling constant and $\Lambda_{\overline{\mathrm{MS}}}$, once power corrections are properly taken in consideration.

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